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COUETTE SLIP FLOW WITH SIDEWAYS PRESSURE GRADIENT FOR NON-NEWTONIAN POWER-LAW FLUID

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An analytical solution of a non-Newtonian power law fluid between parallel plates has been identified for situations where the pressure gradient direction is positioned at a general angle to the vector of the relative motion of the plates. Further assumptions include Navier's condition of slip between the liquid and the plates. Navier's condition, for the border value of its parameter, also describes (Dirichlet's) condition of liquid sticking to the walls. The motivation of the task is mentioned to be the fluid flow in an extruder or a screw press.

Keywords: Couette slip flow, non-Newtonian fluid, analytical solution

Introduction

Literature suggests that Couette flow is the flow of viscous liquid brought about the relative movement of the walls, most typically the flow between parallel plates. In the case of Newtonian fluid, the notion of Couette flow is related especially to Taylor's very profound results [6] who used analytical solutions and their comparisons to the experiments in order to investigate the instability (which bears his name) of liquid flow between co-axial cylindrical walls with the rotating inner cylinder.

In the case of non-Newtonian fluid, any analytical results are scarce and they have been obtained only for "simple" situations (see e.g. [2], [5]). In the case of flow between parallel plates, an analytical solution has been identified for the limitations given below (see e.g. [4]):

1) it is a highly viscous non-Newtonian power-law fluid (see formula (3)), 2) the gap between the plates is constant, the flow is steady state, 3) pressure gradient direction is the same as that of the plate velocity vector (the other plate is at standstill without limitations of generality), 4) the liquid sticks to the walls.

One of the exceptions is the paper by Lawal and Kalyon [1] who instead of 1) assumed the more general Herschel-Bulkley fluid and also considered the liquid slip at the walls. However, limitation 3) still means this is one-dimensional flow. This paper explores a situation that generalizes items 3) and 4). In detail:

the 3generalized) pressure gradient is at a general angle with the plate velocity vector while 4generalized) liquid may slip at the walls as per Navier's condition (4).

Given the condition of 3generalized), this is two-dimensional flow.

The following paragraph deals with the motivation for the investigation of this flow type in relation to the modelling of processes in screw extrudes and presses. The third paragraph shows the mathematical formulation of the problem and the procedure for the isolation of the analytical solution. The fourth paragraph brings the analytical solution of the problem expresses in the dimensionless form. The next paragraph, fifth, sets forth visualization of the results and flow profiles for certain selected situations. The paper ends in the Appendix that describes certain mathematical details.

2. Motivation

Extrusion technology has played an important role in many industries, such as polymer, food, and feed processing, for over 50 years. While processed in the technology, the materials behave as non-Newtonian fluids.

The author of the paper and his colleagues participated in the mathematical modelling of oilseed separation pressing. A procedure has been designed in cooperation with employees of Farmet a.s. for the specification of rheology parameters of oil-solid mixture; the team also participated in the design and evaluation of tests. The procedure is based on the pressing of the oil-solid mixture through a circular slit (see Fig. 1). The pressing screens were manufactured of various slit thickness *H* and length parameters *L*. The principal monitored parameters were the oil-solid mixture flow rate *Q* and the pressure gradient $\Delta p/L$ along the slit. The principal result was that it is not Coulomb friction what is occurring between the oil-solid mixture and the walls, as the pressure gradient would exponentially increase with the slit length; the phenomenon is slip. It was described using the Navier condition (4). The rheology behaviour of the oil-solid mixture was expressed as the behaviour of a non-Newtonian power-law fluid as given in the constitutive formula (3). The relationship (see [8])

$$Q = -\pi D \eta \frac{H^2}{2}\frac{\Delta p}{L} - \pi D \frac{2}{K^{1/n}}\frac{n}{2n+1}\left(\frac{H}{2}\right)^{\frac{2n+1}{n}}\left(\frac{\Delta p}{L}\right)^{1/n}, \tag{1}$$

is easy to deduce. The conducted test as well as data from literature (see e.g. [7]) indicate that the index *n* for rapeseed oil-solid mixture may amount to as much as ≈0.12. As such, it is a highly non-Newtonian fluid. Therefore, the given formula (1) may be largely inaccurate if one of the walls of the assumed circular slit is rotating (the case of the end nozzle of the screw press or the case of its insert).

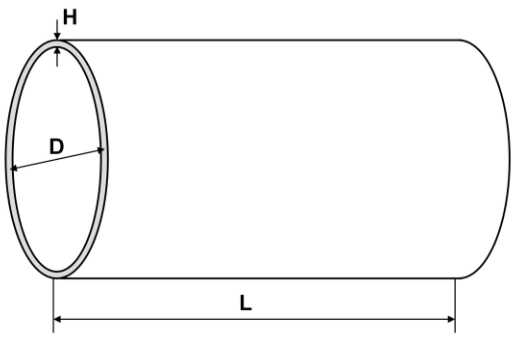

Fig. 1: Circular slit

The situation becomes even more complicated in the screw sections of presses or extruders. The traditional theory for the relationship between the flow and pressure gradient is based on the idea of the press being a flattened-out (shallow) channel. The theory contains the linear composition of the so-called drag flow caused by the relative movement of the walls and the flow resulting from the pressure gradient, which is mathematically accurate in the case of Newtonian fluid at a justifiable disregard of the inertia forces (the fluid is “highly viscous”). (“Leakage flow” is also disregarded for the sake of simplicity.) Therefore at the liquid sticking to the walls

$$Q = \frac{\pi}{2} DNbH \cos\varphi - \frac{bH^3}{12\mu} \frac{dp}{dz} \quad , \tag{2}$$

where Q represents the overall volumetric flow, N is the number of revolutions per a unit of time, D is the slit diameter, H is the slit thickness, φ is the screw angle, $b = \pi D \sin\varphi$ is the channel width, $\mu = K$ fluid viscosity.

In the case of the non-Newtonian fluid, the mentioned linear composition of the flows as per (2) becomes inaccurate. It is furthermore necessary to consider the effect of the n, the fluid index (i.e. relationship (3)), which Shirato et al. [3] respect by stating the viscosity in the formula $\mu = nK(\pi DN\cos\varphi / H)^{n-1}$ and include shape factors, which also depend on the parameters of n, H, b, in the formulae of both partial flows.

An analytical (i.e. accurate) solution of a non-Newtonian fluid flow in a shallow channel can be obtained using the procedure given below, for the case of parallel plates. However, the corresponding mathematical procedure is more complicated.

3. Mathematical formulation of the problem

The constituting formula for the non-Newtonian fluid writes as

$$\tau = \mu\gamma \qquad \text{with} \quad \mu = K\gamma^{n-1} \; , \; 0 < n \le 1 \; , \; K > 0 \; , \tag{3}$$

where τ is the shear stress and γ is the size of the velocity gradient. If $n = 1$, the fluid is of the Newtonian type. Navier's condition will be used to describe the liquid slip at the wall

$$\eta\tau = U \quad , \tag{4}$$

where $\eta \ge 0$ is a material constant i.e., slip coefficient which also depends on the nature of the wall. U is the wall slip velocity and τ is the wall shear stress. In the case of $\eta = 0$, the condition (4) describes the typically assumed sticking of the fluid on the wall, i.e. a situation without slip. Assuming the high viscosity of the liquid, the inertia forces will be disregarded.

Also assume that a pressure gradient β is forced between two parallel plates and that one of the plates is moving at velocity V at an angle φ from that gradient. The situation is shown in Figure 2 which also portrays the system of coordinates used herein. Axis z has the direction of the pressure gradient, axis y is perpendicular to the plates and the beginning of the system of coordinates is located between them. The distance between the plates is H.

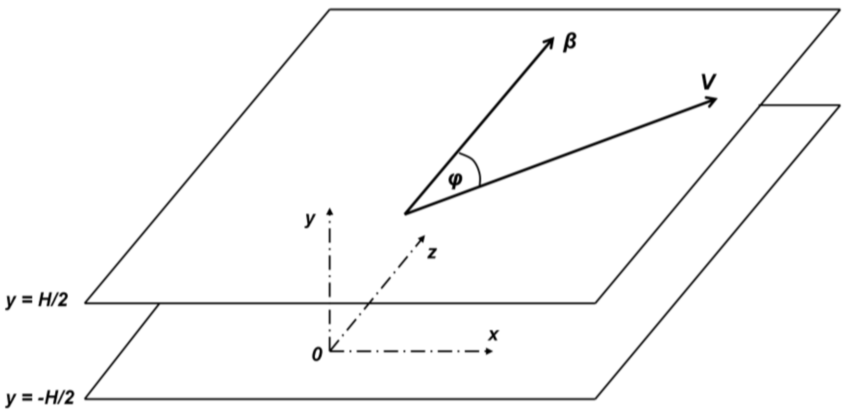

Fig. 2: Situation of two parallel plates

The fluid velocity components in the direction of axes x, y, z shall be denoted, in the order, u, v, w. Deriving from the nature of the task $v = 0$, the components u and w are functions of parameter y only, therefore $u = u(y)$, $w = w(y)$. As such, it is two-dimensional flow. The force balance equations will use the form

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0 \ , \tag{5}$$

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right) = \frac{\partial p}{\partial z} \quad (= \beta) \ , \tag{6}$$

where
$$\mu = K\gamma^{n-1} \ , \ 0 < n \le 1 \ , \ K > 0 \ , \tag{7}$$

with
$$\gamma = \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^{1/2} . \tag{8}$$

The conditions on the walls will be expressed as formulae

$$\eta_- \tau_{xy} = u\left(-\frac{H}{2}\right) \ , \qquad \eta_- \tau_{yz} = w\left(-\frac{H}{2}\right) \ , \tag{9}$$

$$\eta_+ \tau_{xy} = u\left(\frac{H}{2}\right) - V\sin\varphi \ , \qquad \eta_+ \tau_{yz} = w\left(-\frac{H}{2}\right) - V\cos\varphi \ . \tag{10}$$

The paragraph provides a brief description of the solution of task (5)-(10). It is especially by the first integration of equations (5) and (6) using the variable y that formulae for shear stresses are given

$$\tau_{xy} \equiv \mu \frac{\partial u}{\partial y} = A \ , \tag{11}$$

$$\tau_{yz} \equiv \mu \frac{\partial w}{\partial y} = \beta y + B \ , \tag{12}$$

where A and B are integration constants that are unknown as yet. The squaring and addition of relations (11) and (12) gives

$$\mu^2\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = A^2 + (\beta y + B)^2 \ . \tag{13}$$

Therefore, the substitution for μ from (7) and (8) to (11) will be

$$K\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}} \frac{\partial u}{\partial y} = A \ . \tag{14}$$

Use that to express $\left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{A}{K}\right)^{\frac{2}{n-1}} \left|\frac{\partial u}{\partial y}\right|^{\frac{-2}{n-1}} - \left(\frac{\partial u}{\partial y}\right)^2$

and substitute to (13) to obtain

$$\left(\mu \frac{\partial u}{\partial y}\right)^2 \left(\frac{A}{K}\right)^{\frac{2}{n-1}} \left(\frac{\partial u}{\partial y}\right)^{\frac{-2}{n-1}-2} = (\beta y + B)^2 + A^2 \ . \tag{15}$$

Now, use (11): exclude μ from (15) and subsequently calculate

$$\frac{\partial u}{\partial y} = A \left(\frac{1}{K}\right)^{1/n} \left[(\beta y + B)^2 + A^2\right]^{\frac{1-n}{2n}} \ . \tag{16}$$

As (11) also suggests $\mu = A / \left(\frac{\partial u}{\partial y}\right)$, substitution to (12) gives

$$\frac{\partial w}{\partial y} = \left(\frac{1}{K}\right)^{1/n} (\beta y + B)\left[(\beta y + B)^2 + A^2\right]^{\frac{1-n}{2n}} \ . \tag{17}$$

Further integration of relationships (16) and (17) using the variable y obtains the velocity components u and w. To identify the constants A and B, require the conformance to the boundary conditions (9) and (10). This leads to the issue of solving algebraic equations, without derivations and integrals. The following paragraph brings the solution converted into the dimensionless form. The components of volumetric flow per unit of length Q_x, Q_z are obtained by (third) integration by y, this time using the u, w velocity components.

4. Dimensionless formulation of the solution

The flow of non-Newtonian power-law fluid between two parallel plates, as described in the preceding paragraph, depends on the rheologic parameters of the fluid, distance between the plates, on the pressure gradient, plate velocity and the angle between the vectors, also on the slip parameters of the upper and lower plates, i.e. on the parameters given below K, n, H, β, V, φ, η_+, η_-. Use symbols $\xi = \frac{y}{H/2}$, $\bar{u}(\xi) = \frac{u(y)}{V}$, $\bar{w}(\xi) = \frac{w(y)}{V}$, $\bar{Q}_x = \frac{Q_x}{VH}$, $\bar{Q}_z = \frac{Q_z}{VH}$ and assume dimensionless parameters

$$\Omega = \frac{V}{H/2}\left(\frac{2K}{\beta H}\right)^{1/n}, \quad \Gamma_+ = \eta_+ \beta \left(\frac{2K}{\beta H}\right)^{1/n}, \quad \Gamma_- = \eta_- \beta \left(\frac{2K}{\beta H}\right)^{1/n}. \tag{18}$$

Also use dimensionless form of the desired integration constants

$$a = \frac{2A}{\beta H} \ , \qquad b = \frac{2B}{\beta H} \ . \tag{19}$$

Dimensionless solution of the problem (5)-(10) can therefore take the form of

$$\bar{u}(\xi) = \frac{1}{\Omega} a^{2r+2} \left\{ F_r\left(\frac{\xi + b}{a}\right) - F_r\left(\frac{-1+b}{a}\right) \right\} + \frac{\Gamma_-}{\Omega} a \ , \tag{20}$$

$$\overline{w}(\xi) = \frac{1}{\Omega}\,\frac{1}{2r+2}\,a^{2r+2}\left\{G_{r+1}\!\left(\frac{\xi+b}{a}\right) - G_{r+1}\!\left(\frac{-1+b}{a}\right)\right\} + \frac{\Gamma_-}{\Omega}(b-1)\,, \qquad (21)$$

$$\overline{Q}_x = \frac{1}{\Omega}\left\{a^{2r+2}\,\frac{1+b}{2}\left[F_r\!\left(\frac{1+b}{a}\right) - F_r\!\left(\frac{-1+b}{a}\right)\right] - \frac{1}{4r+4}\,a^{2r+3}\left[G_{r+1}\!\left(\frac{1+b}{a}\right) - G_{r+1}\!\left(\frac{-1+b}{a}\right)\right]\right\} + $$
$$+\frac{\Gamma_-}{\Omega}\,a\,,$$

$$\overline{Q}_z = \frac{1}{\Omega}\,\frac{1}{2r+2}\,a^{2r+2}\left\{\frac{a}{2}\left[F_{r+1}\!\left(\frac{1+b}{a}\right) - F_{r+1}\!\left(\frac{-1+b}{a}\right)\right] - G_{r+1}\!\left(\frac{-1+b}{a}\right)\right\} + \frac{\Gamma_-}{\Omega}(b-1)\,,$$

denoting $r = \dfrac{1-n}{2n}$, $G_r(t) = \left(1+t^2\right)^r$, $F_r(t) = \displaystyle\int_0^t G_r(s)\,ds$ for $t > 0$, $F_r(t) = -F_r(-t)$ for $t < 0$. The calculation of the function $F_r(t)$ is covered in the Appendix. Dimensionless integration constants a, b which charakterize dimensionlees shear stresses will be obtained by solving non-linear (algebraic) equations

$$a^{2r+2}\left\{F_r\!\left(\frac{1+b}{a}\right) - F_r\!\left(\frac{-1+b}{a}\right)\right\} = \Omega\sin\varphi - \left(\Gamma_+ + \Gamma_-\right)a\,, \qquad (22)$$

$$\frac{1}{2r+2}\,a^{2r+2}\left\{G_{r+1}\!\left(\frac{1+b}{a}\right) - G_{r+1}\!\left(\frac{-1+b}{a}\right)\right\} = \Omega\cos\varphi - \left(\Gamma_+ + \Gamma_-\right)b + \left(\Gamma_+ - \Gamma_-\right)\,. \qquad (23)$$

5. Special situations and velocity profiles

As an illustration, we describe the particular situation of $\eta_+ = \eta_- = 0$ (i.e. $\Gamma_+ = \Gamma_- = 0$) when no slip occurs. Then dimensionless velocity profiles depend only on the power index n, the angle φ and the dimensionlees number Ω. They are depicted for indices n = 0.2, 0.5 and for various values of Ω and φ in Fig. 3. Provided $\varphi = 0$ or π we have from the equation (22) that $a = 0$ and $\overline{u}(\varsigma) \equiv 0$. Asymptotic behaviour of functions F_r and G_r for the limit case $a \to 0$ are given in Appendix. On the other side it is $b = 0$ from (23) for $\varphi = \pi/2$.

Conclusion

An analytical solution of Couette slip flow between two parallel plates is derived for the non-Newtonian power-law fluid (see (3)). The flow is two-dimensional one if the pressure gradient contains the non-vanishing angle φ (see Fig. 2) with the wall velocity vector. The Navier's condition (4) is used to describe the fluid slip at walls. Profiles of dimensionlles fluid velocity (20), (21) then depend only on the index n, the angle φ and dimensionlees numbers Ω, Γ_+, Γ_- (see (18)). Of course, we need to find parameters a, b solving systems of two non-linear algebraic equations (22), (23). The parameters a and b characterize dimensionlees shear stresses.

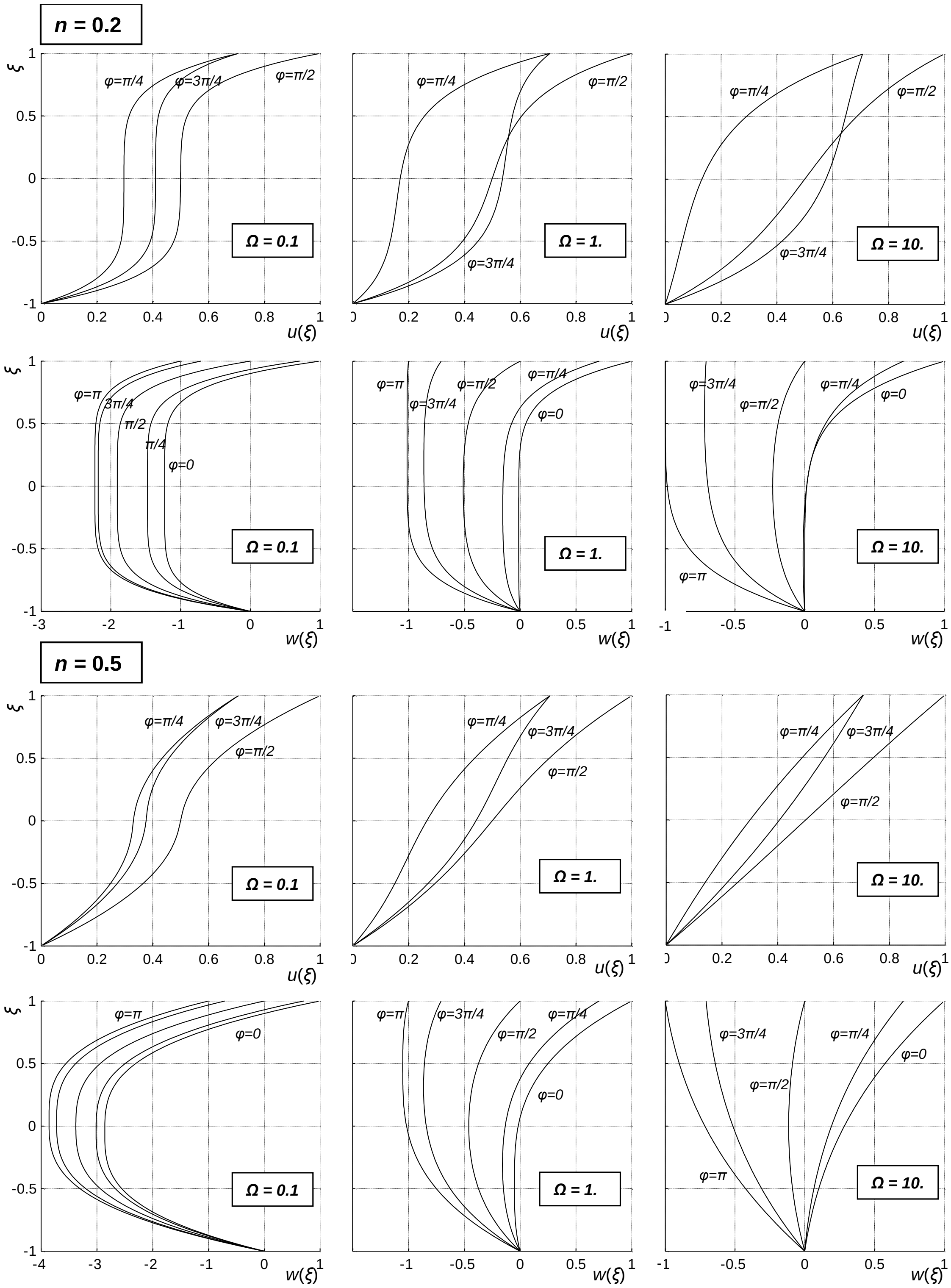

Fig. 3: Dimensionless velocity profiles for power indices n = 0.2, 0.5 and for various values of Ω and φ, $\Gamma_{+} = \Gamma_{-} = 0$

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Appendix

Using the binomial theorem for a non-integer exponent and integrating behind summation sign we can write

$$F_r(X) = \sum_{k=0}^{\infty} (-1)^k \frac{(-r)_k}{k!} \frac{1}{2k+1} X^{2k+1} \qquad \text{for } 0 \le X \le 1 ,$$

$$F_r(X) = \sum_{k=0}^{\infty} (-1)^k \frac{(-r)_k}{k!} \frac{1}{2k+1} + \sum_{k=0}^{\infty} c_k \qquad \text{for } 1 < X$$

with $\quad c_k = \dfrac{(-r)_k}{k!} \dfrac{(-1)^k}{2r-2k+1} \left(X^{2r+2k+1} - 1 \right) \quad$ if $\;$ 2r-2k+1 $\neq 0$

or $\quad c_k = \dfrac{(-r)_k}{k!} (-1)^k \ln X \quad$ if 2r-2k+1 = 0.

Here $(-r)_k = -r\,(-r+1) \ldots (-r+k-1)$ is the Pachhammer symbol. Further,

$$a^{2r+1} F_r\left(\frac{c}{a}\right) \to \operatorname{sgn}(c) \frac{|c|^{2r+1}}{2r+1} \qquad \text{for } \; a \to 0_+ \; ,$$

$$a^{2r} G_r\left(\frac{c}{a}\right) = \left(c^2 + a^2\right)^r \quad \text{for } \; a > 0 \; .$$

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